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A simplex of bound entangled multipartite qubit states

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We construct a simplex for multipartite qubit states of even number n of qubits, which has the same geometry concerning separability, mixedness, kind of entanglement, amount of entanglement and nonlocality as the bipartite qubit states. We derive the entanglement of the class of states which can be described by only three real parameters with the help of a multipartite measure for all discrete systems. We prove that the bounds on this measure are optimal for the whole class of states and that it reveals that the states possess only n–partite entanglement and not e.g. bipartite entanglement. We then show that this n–partite entanglement can be increased by stochastic local operations and classical communication to the purest maximal entangled states. However, pure n–partite entanglement cannot be distilled, consequently all entangled states in the simplex are n–partite bound entangled. We study also Bell inequalities and find the same geometry as for bipartite qubits. Moreover, we show how the (hidden) nonlocality for all n–partite bound entangled states can be revealed.

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I. INTRODUCTION

Entanglement is at the heart of the quantum theory. It is the source of several new applications as quantum cryptography or a possible quantum computer. In recent years by studying higher dimensional quantum systems and/or multipartite systems one realizes that different aspects of the entanglement feature arise. They may have new applications such as multiparty cryptography.

In this paper we contribute to the classification of entanglement in a twofold way, i.e. which kind of entanglement a certain class of multipartite qubit states possesses by using the multipartite measure proposed in Ref. [1] and whether this kind of entanglement can be distilled. Our results suggest that one can distinguish for multipartite systems between different possibilities.

The class of states we analyze are a generalization of the class of states which form the well–known simplex for bipartite qubits (Sec. II), i.e. all locally maximally mixed states [2, 3]. We make an obvious generalization and find for states composed of an even number of qubits n an analogous simplex, i.e. this class of states shows the same geometry concerning positivity, mixedness, separability and entanglement (Sec. III). Further the used multipartite measure [1] reveals that the kind of entanglement possessed is only a n–partite entanglement where n is the number of qubits involved. The vertex states of the simplex are represented in the bipartite case by the well known Bell states, for $n>2$ they are equivalent to the generalized smolin states proposed by Ref. [4, 6, 7, 8].

Then we discuss the distillability of the entangled states and find states for which the n–partite entanglement can be increased by a protocol only based on copy states and stochastic local operations and classical communications (LOCC). We show that the state is not distillable for any subset of parties and hence bound entangled, however, the n–partite entanglement can be enhanced to reach the maximal possible purity and n–partite entanglement within the class of states under investigation, i.e. the vertex states. For a subset of these states it has been shown that they allow for quantum information concentration ,e.g. Ref. [4, 5], so we suggest that it might still be advantageous to enhance the n–partite bound entangled states for some applications.

Last but not least, in Sec. VI we address to the question which of the simplex states violate the generalized Bell inequality which was shown to be optimal in this case and draw its geometrical picture, Fig. 4.

II. THE SIMPLEX FOR BIPARTITE QUBITS

A single qubit state ω lives in a two dimensional Hilbert space, i.e. $\mathcal{H}\equiv\mathbb{C}^2$, and any state can be decomposed into the well known Pauli matrices σ_i

$$\omega \;=\; \frac{1}{2}\,(\mathbb{1}_2 + n_i\,\sigma_i)$$

with the Bloch vector components $\vec{n}\in\mathbb{R}^3$ and $\sum_{i=1}^3 n_i^2 = |\vec{n}|^2 \leq 1$. For $|\vec{n}|^2<1$ the state is mixed (corresponding to $\mathrm{Tr}\,\omega^2<1$) whereas for $|\vec{n}|^2=1$ the state is pure ($\mathrm{Tr}\,\omega^2=1$).

The density matrix of 2–qubits ρ on $\mathbb{C}^2\otimes\mathbb{C}^2$ is usually obtained by calculating its elements in the standard product basis, i.e. $|00\rangle,|01\rangle,|10\rangle,|11\rangle$. Alternatively, we can write any 2–qubit density matrix in a basis of 4×4 matrices, the tensor products of the identity matrix $\mathbb{1}_2$ and the Pauli matrices,

$$\rho = \frac{1}{4}\,(\mathbb{1}_2\otimes\mathbb{1}_2 + a_i\,\sigma_i\otimes\mathbb{1}_2 + b_i\,\mathbb{1}_2\otimes\sigma_i + c_{ij}\,\sigma_i\otimes\sigma_j)$$

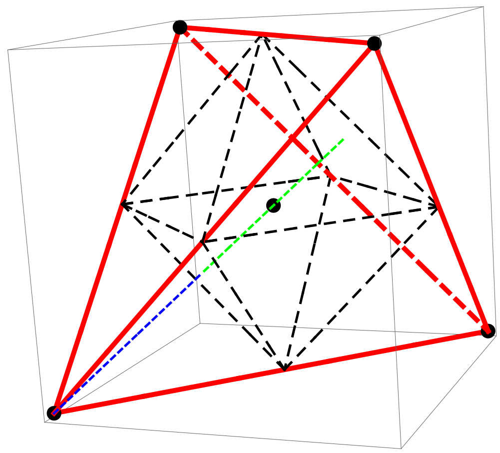

FIG. 1: (Color online) Here the geometry of the state space of even number of qubits is visualized. Each state is represent by a triple of three real numbers, $\vec{c}$, Eq. (1). The four black dots in the vertices of the cube represent four orthogonal "vertex" states. In the case of two qubits these are the four maximally entangled Bell states $\psi^\pm, \phi^\pm$ and for higher n they are equally mixtures of $2^n/4$ GHZ–states. The positivity condition forms a tetrahedron (red) with the four "vertex" states and the totally mixed state in the origin (black dot in the middle). All separable states are represented by points inside and at the surface of the octahedron (dashed object). The dashed line represents for $n=2$ the Werner states and for $n>2$ the generalized Smolin states (becoming separable when blue changes into green).

with $a_i, b_i, c_{ij} \in \mathbb{R}$. The parameters a_i, b_i are called *local* parameters as they determine the statistics of the reduced matrices, i.e. of Alice's or Bob's system. In order to obtain a geometrical picture one considers in the following only states where the local parameters are zero ($\vec{a} = \vec{b} = \vec{0}$), i.e., the set of all locally maximally mixed states, $Tr_A(\rho) = Tr_B(\rho) = \frac{1}{2}\mathbb{1}_2$ (see also Ref. [2, 3]).

A state is called separable if and only if it can be written in the form $\sum_i p_i\, \rho_i^A \otimes \rho_i^B$ with $p_i \geq 0, \sum p_i = 1$, otherwise it is entangled. As the property of separability does not change under local unitary transformation and classical communication (LOCC) the states under consideration can be written in the form [2]

$$\rho \;=\; \frac{1}{4}\left(\mathbb{1}_2 \otimes \mathbb{1}_2 + c_i\, \sigma_i \otimes \sigma_i\right) ,$$

where the c_i are three real parameters and can be considered as a vector $\vec{c}$ in Euclidean space. Differently stated, for any locally maximally mixed state ρ the action of two arbitrary unitary transformations $U_1 \otimes U_2$ can via the homomorphism of the groups $SU(2)$ and $SO(3)$ be related to unique rotations $O_1 \otimes O_2$. Thus the correlation matrix $c_{ij}\, \sigma_i \otimes \sigma_j$ can be chosen such that the matrix c_{ij} gets via singular value decomposition diagonal. Therefore, three real numbers combined to a vector $\vec{c}$ can be taken as an representative of the state itself.

In Fig. 1 we draw the 3–dimensional picture, where each point $\vec{c}$ corresponds to a locally maximally mixed state ρ. The origin $\vec{c} = \vec{0}$ corresponds to the totally mixed state, i.e. $\frac{1}{4}\mathbb{1}_2 \otimes \mathbb{1}_2$. The only pure states in the picture are given by $|\vec{c}|^2 = 3$ and represent the four maximally entangled Bell states $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}\{|01\rangle \pm |10\rangle\}, |\phi^\pm\rangle = \frac{1}{\sqrt{2}}\{|00\rangle \pm |11\rangle\}$, which are located in vertices of the cube. The planes spanned by these four points are equivalent to the positivity criterion of the state ρ. Therefore, all points inside the tetrahedron represent the state space.

It is well known that density matrices which have at least one negative eigenvalue after partial transpose (PT) are entangled. The inversion of the argument is only true for systems with $2 \otimes 2$ and $2 \otimes 3$ degrees of freedom. PT corresponds to a reflection, i.e. $c_2 \rightarrow -c_2$ with all other components unchanged. Thus all points inside and at the surface of the octahedron represent all separable states in the set. Of course, one can always make the transformation $\vec{c} \longrightarrow -\vec{c}$, thus one obtains a mirrored tetrahedron, spanned by the four other vertices of the cube. Clearly, the intersection of the these two tetrahedrons contain all states which have positive eigenvalues after the action of PT.

In Ref. [9, 10, 11] a generalization to higher dimensional bipartite states is considered and a so called magic simplex for qudits is obtained. Here the class of all locally maximally mixed states have to be reduced in order to obtain this generalized simplex. Already for bipartite qutrits many new symmetries arise and regions of bound entanglement can be found (see also Refs. [12, 13, 14, 15, 16]).

We also want to generalize the simplex of bipartite qubits, however, in our case we increase the number of qubits.

III. A SIMPLEX FOR n–PARTITE QUBIT STATES

Assume we have n qubits. Then a generalization can be written as

$$\begin{aligned} \rho \;&=\; \frac{1}{2^n}\left(\mathbb{1} + \sum c_i\, \sigma_i \otimes \sigma_i \otimes \cdots \otimes \sigma_i\right) \\ &:= \frac{1}{2^n}\left(\mathbb{1} + \sum c_i\, \sigma_i^{\otimes n}\right) \qquad (1) \end{aligned}$$

Obviously, for this generalization we follow the strategy to set the local parameters of all subsystems j, $Tr_{1,2,\ldots,j-1,j+1,\ldots,n}(\rho)$, to zero, as well as the parameters shared by two parties j,k, $Tr_{1,2,\ldots,j-1,j+1,\ldots,k-1,k+1,\ldots,n}(\rho)$, zero and so on until $n-1$ zero.

Again the state can be represented by a three dimensional vector $\vec{c}$. For $n=3$ the positivity condition $\rho \geq 0$ requires

$$|\vec{c}|^2 \leq 1 \; . \qquad (2)$$

This turns out to be the case for all odd numbers of qubits involved.

For even numbers of qubits the positivity condition $\rho \geq 0$ requires that the vector is within the following four planes [30]:

$$1 + \vec{c}\cdot\vec{n}^{\,(i)} \;\geq\; 0 \tag{3}$$
$$\text{with} \quad \vec{n}^{\,(i)} \;=\; \begin{pmatrix} -1 \\ +1 \\ +1 \end{pmatrix}, \begin{pmatrix} +1 \\ -1 \\ +1 \end{pmatrix}, \begin{pmatrix} +1 \\ +1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

These conditions are exactly the same ones as for the two qubit case $n = 2$, i.e. the four above planes form the magic tetrahedron.

The purity $Tr(\rho^2)$ gives $\frac{1}{2^n}(1 + |\vec{c}|^2)$, thus the states with $|\vec{c}|^2 = 3$ are the purest states of the class of states under investigation and are located in the vertices of the tetrahedron. Note with increasing n the percentage of purity decreases, i.e. only for $n = 2$ the vertices present pure states. Further analysis of these vertex states follows later.

Now we want to investigate if the separability condition also for $n > 2$ corresponds to the octahedron. The partial transpose of one qubit ($PT_{\text{one qubit}}$) changes the sign in front of the $\sigma_2^{\otimes n}$ matrix, i.e. the y–component of the vector $\vec{c}$ changes sign. Therefore the states under investigation are entangled by the necessary but not sufficient (one qubit) Peres criterion

$$\begin{aligned} \text{n=3,5,\ldots:} &\quad |\vec{c}\,|^2 \leq 1 \\ \text{n=2,4,\ldots:} &\quad 1 - \vec{c}\cdot\vec{n}^{\,(i)} \;\leq\; 0\;. \end{aligned} \tag{4}$$

Taking the partial transpose of two, four, …qubits changes two, four,…times the sign and consequently one obtains the positivity criterion (3). Taking the partial transpose of odd qubits is equivalent to $PT_{\text{one qubit}}$.

For even number of qubits the above Peres criterion implies a mirrored tetrahedron, analogously to the bipartite case, however, we do not know if the intersection, the octahedron, contains only separable states. For odd numbers of qubits the situation is different and we will not investigate it further.

Now two questions arise, firstly, are all states represented by the octahedron separable and, secondly, what kind of entanglement does this class of states possess?

Let us tackle the second question first. To analyze our generalized states ρ further we use the multipartite entanglement measure for all discrete systems introduced by Ref. [1]. The main idea is that the information content of any n–partite quantum system of arbitrary dimension can be separated in the following form:

$$\underbrace{I(\rho) + R(\rho)}_{\text{single property}} + \underbrace{E(\rho)}_{\text{entanglement}} \;=\; n \tag{5}$$

where

$$I(\rho) \;:=\; \sum_{s=1}^{n} \underbrace{\mathcal{S}_s^2(\rho)}_{\text{single property of subsystem s}} \tag{6}$$

contains all locally obtainable information (i.e. obtainable information a party can measure on its particle) and $E(\rho)$ contains all information encoded in entanglement and $R(\rho)$ is the complementing missing information, it is due to a classical lack of knowledge about the quantum state. The total amount of entanglement $E(\rho)$ can be separated into m–flip concurrences by rewriting the linear entropy of all subsystems in an operator sum, thus one obtains

$$\begin{aligned} E(\rho) \;:=\; & \underbrace{\mathbf{C}^2_{(2)}(\rho)}_{\text{two flip concurrence}} + \underbrace{\mathbf{C}^2_{(3)}(\rho)}_{\text{three flip concurrence}} + (\ldots) \\ & + \underbrace{\mathbf{C}^2_{(n)}(\rho)}_{\text{n-flip concurrence}}\;. \end{aligned} \tag{7}$$

These m–flip concurrences are useful for two reason: firstly, one can obtain bounds on the operators and thus handle mixed states and secondly, the authors of Ref. [1] showed (for three qubits) that the m–flip concurrences can be reordered such that they give the m–partite entanglement, which in addition coincides with the m–separability [17].

Here we extend their result for the states under investigation. Due to high symmetry of the class of states under investigation the bounds of the m–partite entanglement can be computed and herewith we can reveal the following substructure of total entanglement $E(\rho)$

$$\begin{aligned} E(\rho) \;=\; & \underbrace{E_{(2)}(\rho)}_{\text{bipartite entanglement}} + \underbrace{E_{(3)}(\rho)}_{\text{tripartite entanglement}} \\ & + \cdots + \underbrace{E_{(n)}(\rho)}_{n\text{–partite entanglement}} \end{aligned} \tag{8}$$

with the sub–substructure

$$\begin{aligned} E_{(2)}(\rho) \;&=\; E_{(12)}(\rho) + E_{(13)}(\rho) + \cdots + E_{(1n)}(\rho) \\ &\quad + E_{(23)}(\rho) + \cdots + E_{(2n)}(\rho) + \cdots + E_{(n-1,n)}(\rho) \\ E_{(3)}(\rho) \;&=\; E_{(123)}(\rho) + \cdots + E_{(n-2,n-1,n)}(\rho) \\ \ldots \;&=\; \ldots \\ E_{(n)}(\rho) \;&=\; E_{(12\ldots n)}(\rho)\;. \end{aligned} \tag{9}$$

We find that for the states under investigation the only non–vanishing entanglement is the n–partite entanglement and it derives to (for details next Sec. IV)

$$\begin{aligned} E_{(n)} \;=\; E_{12\ldots n} \;=\; & X \max\big[0,\; \frac{1}{2}\max\Big[-1 + \vec{c}\cdot\vec{n}^{\,(1)}, \\ & -1 + \vec{c}\cdot\vec{n}^{\,(2)}, -1 + \vec{c}\cdot\vec{n}^{\,(3)}, -1 + \vec{c}\cdot\vec{n}^{\,(4)}\,\big]\Big]^2\;, \end{aligned} \tag{10}$$

where $X = 1$ except for bipartite qubits then it is $X = 2$ (the reason of this difference is explained later). Hence, we find the same condition for being entangled as given by the one qubit Peres criterion.

Now, if these bounds are exact also for $n > 2$, then all states represented by the octahedron are separable. Indeed, it turns out that this is the case. We give the proof of separability separately in the appendix.

In summary, we have found for even number of qubits the same geometry as in the case of bipartite qubits, also depicted by Fig. 1. Moreover, we have shown that the multipartite entanglement measure proposed by Ref. [1] works tightly as the bounds are exact and it reveals only n–partite entanglement. Let us discuss this result more carefully.

For the purest states, $|\vec{c}|^2 = 3$, located in the vertices of the tetrahedron, the maximal n–partite entanglement derives to $E_{(n)} = 1$ except for $n = 2$ it is $E_{(n)} = 2$. Thus the amount of entanglement for $n > 2$ is independent of the number of qubits involved. The reason for the difference can be found in the information content of a multipartite system, Eq. (5). The maximal entanglement of a n–partite state is n. This is the case if and only if the local obtainable information of all subsystems is zero and the classical lack of knowledge of the quantum state is also zero, i.e. the total state is pure. For bipartite qubits, $n = 2$, the vertex states are the Bell states, which have maximal entanglement 2 whereas there locally obtainable information S is zero as well as the lack of classical knowledge about the quantum state $R = 0$.

By construction for $n > 2$ we set the locally obtainable information S of all subsystems zero, however, also all possible locally obtainable information shared by two, three, $\ldots$, $n-1$ parties is set to zero; obviously this is not compatible with being maximally entangled. The information content for $n > 2$ is given by

$$n \;=\; E_n + R \;=\; 1 + R\,, \tag{11}$$

and consequently the lack of classical knowledge is nonzero, i.e. $R = n - 1$. Differently stated for $n = 4$, any party has the trace state as well as any two parties and any three parties share the trace state, therefore $R = 3$.
Remark: The local information $S_s(\rho)$ of one subsystem s is nothing else than Bohr's quantified complementarity relation [18, 19, 20], with its well known physical interpretation in terms of predictability and visibility (coherence). One can extend this concept for two parties sharing a state, then their (bi–)local information of total multipartite system can be defined in similar way and is complemented by the mixedness of the shared bipartite system. Again this (bi–)local information is only obtainable if and only if the state is not the trace state.

Coming back to the simplex geometry we see that the closer we get to the origin the more the amount of entanglement reduces by increasing the amount of classical uncertainty R only.

For bipartite qubits the vertex states $|\vec{c}|^2 = 3$ are the four Bell states. For n qubits we find for $|\vec{c}|^2 = 3$ also four unitary equivalent states, however, they are no longer pure. For $n = 4$ the state is a equally weighted mixture of four $|GHZ\rangle$ states: Starting with one GHZ–state, e.g.

$$|GHZ\rangle \;=\; \frac{1}{\sqrt{2}}\big\{|0000\rangle + |1111\rangle\big\} \tag{12}$$

one obtains another representation by applying two flips, i.e. $\mathbb{1}\otimes\mathbb{1}\otimes\sigma_x\otimes\sigma_x$, then applying on the new GHZ–state representation the operator $\mathbb{1}\otimes\sigma_x\otimes\sigma_x\otimes\mathbb{1}$ and onto that new GHZ–state representation the operator $\sigma_x\otimes\sigma_x\otimes\mathbb{1}\otimes\mathbb{1}$ gives the last GHZ–state representation. The other three vertex states are obtained by applying only one Pauli matrix. For $n = 6$ we have 2^6 GHZ–states where $2^6/4$ GHZ–states equally mix for one vertex state.
Remark: The same symmetry we find for the bipartite qubit case, one Bell states is mapped into another by one Pauli matrix, however, applying two Pauli matrices maps a Bell state onto itself, therefore we have no mixture of different maximally entangled states.

In the next section we give the detailed calculation of the measure and in the following section we investigate the question whether the entangled states are bound entangled and if in what sense their entanglement is bound. In particular we discuss what it means that the substructure revealed by the measure shows only n–partite entanglement.

IV. DERIVATION OF THE MULTIPARTITE MEASURE FOR THE SIMPLEX STATES

In Ref. [1] a multipartite measure for multidimensional systems as a kind of generalization of Bohr's complementarity relation was derived. Here, we give explicitly the results for $n = 2$ and $n = 4$ expressed in the familiar Pauli matrix representation

It is well known that to compute concurrence introduced by Hill and Wootters [21] one has to consider

$$\rho\;(\sigma_y\otimes\sigma_y)\;\rho^*\;(\sigma_y\otimes\sigma_y) \tag{13}$$

where the complex conjugation is taken in computational basis. The concurrence is then given by the formula

$$\mathcal{C} \;=\; \max\big\{0, 2\max\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\} - (\lambda_1+\lambda_2+\lambda_3+\lambda_4)\big\} \tag{14}$$

where the λ_i's are the square roots of the eigenvalues of the above matrix. To obtain the information content we have to multiply this measure by two.

The first observation in Ref. [1] is that the linear entropy, $M(\rho) = \frac{2}{3}\big(1 - Tr(\rho^2)\big)$, can be rewritten by operators. This means e.g. for any pure 4 qubit state

$$|\psi\rangle \;=\; \sum_{i,j,k,l=0}^{1} a_{ijkl}\;|ijkl\rangle\,, \tag{15}$$

the linear entropy of one subsystem can be written as

$$
\begin{aligned}
&M^2(Tr_{234}|\psi\rangle\langle\psi|) = M^2(\rho_1) = \\
&\quad \sum_{k,l=0}^{1} \sum_{\{i_1\neq i'_1\};\{i_2\neq i'_2\}} \left|\langle\psi|(\sigma_x\otimes\sigma_x\otimes\mathbb{1}\otimes\mathbb{1})\,(|i_1\,i_2\,k\,l\rangle\langle i_1\,i_2\,k\,l| - |i'_1\,i'_2\,k\,l\rangle\langle i'_1\,i'_2\,k\,l|)\,|\psi^*\rangle\right|^2 \\
&+ \sum_{k,l=0}^{1} \sum_{\{i_1\neq i'_1\};\{i_3\neq i'_3\}} \left|\langle\psi|(\sigma_x\otimes\mathbb{1}\otimes\sigma_x\otimes\mathbb{1})\,(|i_1\,k\,i_3\,l\rangle\langle i_1\,k\,i_3\,l| - |i'_1\,k\,i'_3\,l\rangle\langle i'_1\,k\,i'_3\,l|)\,|\psi^*\rangle\right|^2 \\
&+ \sum_{k,l=0}^{1} \sum_{\{i_1\neq i'_1\};\{i_3\neq i'_3\}} \left|\langle\psi|(\sigma_x\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_x)\,(|i_1\,k\,l\,i_4\rangle\langle i_1\,k\,l\,i_4| - |i'_1\,k\,l\,i'_3\rangle\langle i'_1\,k\,l\,i'_4|)\,|\psi^*\rangle\right|^2 \\
&+ \sum_{k,l=0}^{1} \sum_{\{i_2\neq i'_2\};\{i_3\neq i'_3\}} \left|\langle\psi|(\mathbb{1}\otimes\sigma_x\otimes\sigma_x\otimes\mathbb{1})\,(|k\,i_2\,i_3\,l\rangle\langle k\,i_2\,i_3\,l| - |k\,i'_2\,i'_3\,l\rangle\langle k\,i'_2\,i'_3\,l|)\,|\psi^*\rangle\right|^2 \\
&+ \sum_{k,l=0}^{1} \sum_{\{i_2\neq i'_2\};\{i_4\neq i'_4\}} \left|\langle\psi|(\mathbb{1}\otimes\sigma_x\otimes\mathbb{1}\otimes\sigma_x)\,(|k\,i_2\,l\,i_4\rangle\langle k\,i_2\,l\,i_4| - |k\,i'_2\,l\,i'_4\rangle\langle k\,i'_2\,l\,i'_4|)\,|\psi^*\rangle\right|^2 \\
&+ \sum_{k,l=0}^{1} \sum_{\{i_3\neq i'_3\};\{i_4\neq i'_4\}} \left|\langle\psi|(\mathbb{1}\otimes\mathbb{1}\otimes\sigma_x\otimes\sigma_x)\,(|k\,l\,i_3\,i_4\rangle\langle k\,l\,i_3\,i_4| - |k\,l\,i'_3\,i'_4\rangle\langle k\,l\,i'_3\,i'_4|)\,|\psi^*\rangle\right|^2 \\
&+ \sum_{k}^{1} \sum_{\{i_1\neq i'_1\};\{i_2\neq i'_2\};\{i_3\neq i'_3\}} \left|\langle\psi|(\sigma_x\otimes\sigma_x\otimes\sigma_x\otimes\mathbb{1})\,(|i_1\,i_2\,i_3\,k\rangle\langle i_1\,i_2\,i_3\,k| - |i'_1\,i'_2\,i'_3\,k\rangle\langle i'_1\,i'_2\,i'_3\,k|)\,|\psi^*\rangle\right|^2 \\
&+ \sum_{k=0}^{1} \sum_{\{i_1\neq i'_1\};\{i_2\neq i'_2\};\{i_4\neq i'_4\}} \left|\langle\psi|(\sigma_x\otimes\sigma_x\otimes\mathbb{1}\otimes\sigma_x)\,(|i_1\,i_2\,k\,i_4\rangle\langle i_1\,i_2\,k\,i_4| - |i'_1\,i'_2\,k\,i'_4\rangle\langle i_1\,i'_2\,k\,i'_4|)\,|\psi^*\rangle\right|^2 \\
&+ \sum_{k=0}^{1} \sum_{\{i_1\neq i'_1\};\{i_3\neq i'_3\};\{i_4\neq i'_4\}} \left|\langle\psi|(\sigma_x\otimes\mathbb{1}\otimes\sigma_x\otimes\sigma_x)\,(|i_1\,k\,i_3\,i_4\rangle\langle i_1\,k\,i_3\,i_4| - |i'_1\,k\,i'_3\,i'_4\rangle\langle i'_1\,k\,i'_3\,i'_4|)\,|\psi^*\rangle\right|^2 \\
&+ \sum_{k=0}^{1} \sum_{\{i_2\neq i'_2\};\{i_3\neq i'_3\};\{i_4\neq i'_4\}} \left|\langle\psi|(\mathbb{1}\otimes\sigma_x\otimes\sigma_x\otimes\sigma_x)\,(|k\,i_2\,i_3\,i_4\rangle\langle k\,i_2\,i_3\,i_4| - |k\,i'_2\,i'_3\,i'_4\rangle\langle k\,i'_2\,i'_3\,i'_4|)\,|\psi^*\rangle\right|^2 \\
&+ \sum_{\{i_1\neq i'_1\};\{i_2\neq i'_2\};\{i_3\neq i'_3\};\{i_4\neq i'_4\}} \left|\langle\psi|(\sigma_x\otimes\sigma_x\otimes\sigma_x\otimes\sigma_x)\,(|i_1\,i_2\,i_3\,i_4\rangle\langle i_1\,i_2\,i_3\,i_4| - |i'_1\,i'_2\,i'_3\,i'_4\rangle\langle i'_1\,i'_2\,i'_3\,i'_4|)\,|\psi^*\rangle\right|^2
\end{aligned}
\tag{16}
$$

where e.g. $\{i_1\}\neq\{i'_1\},\{i_2\}\neq\{i'_2\}$ means that the set of indexes are not the same, i.e. the sum is taken over

$$
\begin{aligned}
\{i_1,i_2;i'_1,i'_2\} \;=\; &\{0,1;0,0\},\{0,0;0,1\},\{0,1;1,0\},\{0,0;1,1\},\{1,1;0,0\},\{1,0;0,1\},\{1,1;0,0\},\{1,0;0,1\},\\
&\{0,0;1,0\},\{1,0;0,0\},\{0,0;1,1\},\{1,0;0,1\},\{0,1;1,0\},\{1,1;0,0\},\{0,1;1,1\},\{1,1;0,1\}\,.
\end{aligned}
\tag{17}
$$

Likewise the linear entropies for the other subsystem can be derived, i.e. separated in terms where the flip operator σ_x is applied two, three or four times. It is well known that for pure states the sum over the entropies of all reduced density matrices is an entanglement measure, therefore using the linear entropy we get the following entanglement measure

$$
E(|\psi\rangle) := \sum_{s=1}^{4} M^2(\rho_s) \;=\; \sum_{m=2}^{4} \left(C^m(\psi)\right)^2\,, \tag{18}
$$

where $(C^m)^2$ is the sum of all terms of all reduced matrices which contain m–flip operators. These quantities where called (squared) m–concurrences, because they play a similar role as Wootters concurrence.

For mixed states ρ the infimum of all possible decompositions is an entanglement measure

$$
E(\rho) \;=\; \inf_{p_i,|\psi_i\rangle} \sum_{p_i,|\psi_i\rangle} p_i E(|\psi_i\rangle)\,. \tag{19}
$$

The problem of the whole entanglement theory is that this infimum can in general not be calculated. Now we bring the operator representation of the linear entropy into the game, because for operators upper bounds can be obtained.

Lets start with the calculation of the 4-flip concurrence $C^{(4)}$, which is the sum of all terms containing 4-flips of the entropies of all reduced matrices, i.e.

$$\left(C^{(4)}(\rho)\right)^2 = \inf_{p_i,|\psi_i\rangle} \sum_{p_i,|\psi_i\rangle} p_i \left(C^{(4)}(\psi_i)\right)^2 \qquad (20)$$

As shown in Ref. [1] one can derive bounds on the above expression for any m–flip concurrence by defining, in an analogous way to Hill and Wootters flip density matrix [21], the m–flip density matrix:

$$\begin{aligned}\widetilde{\rho}_s^m = O_s(|\{i_n\}\rangle\langle\{i_n\}| - |\{i'_n\}\rangle\langle\{i'_n\}|)\, \rho^* \cdot \\ \cdot\, O_s(|\{i_n\}\rangle\langle\{i_n\}| - |\{i'_n\}\rangle\langle\{i'_n\}|) \qquad (21)\end{aligned}$$

and calculating the λ_m^s's which are the squared roots of the eigenvalues of $\widetilde{\rho}_s^m \rho$. The bound $B^{(m)}$ of the m-flip concurrence $C^{(m)}$ is then given by

$$B^m(\rho) := \left(\sum_s \mathbf{max}\left[0, 2\mathbf{max}\left(\{\lambda_m^s\}\right) - \sum\{\lambda_m^s\}\right]^2\right)^{\frac{1}{2}} \qquad (22)$$

From Eq. (16) we see that for the 4–flip concurrence of subsystem ρ_1 four different operators occur, thus we have in total 16 different operators listed in the appendix VII.

Inserting our class of states we find that for each operator $\mathcal{O}^s$ the eigenvalues are the same, i.e. one obtains 8 zeros and the remaining four eigenvalues are exactly equivalent to the Peres criterion Eq. (4).

The same procedure has to be applied to calculate the 3–flip concurrence and the 2–flip concurrence. As can be seen from Eq. (16) here the unity and σ_z matrix is involved which lead to no contribution for the states under investigation. Remember, that they are mixtures of the vertex states, which are equal mixtures of such GHZ–states which differ by two flips.

Therefore, the total entanglement is given by the $C^{(4)}$ concurrence only and is a 4–partite entanglement. For $n = 6, 8, \ldots$ the scenario is the same, because of the same underlying symmetry.

In the Appendix we show that all states not detected by the measure are separable, thus the bounds are optimal and therefore the measure detects all bound entangled states.

V. ARE THE ENTANGLED STATES BOUND ENTANGLED?

In Refs. [4, 6, 7, 8] the special states $c = c_1 = -c_2 = c_3$ for $n > 2$, which were named generalized Smolin states (for $n = 2$ these states are the Werner states), are investigated and they show that for $1 \geq c > \frac{1}{3}$ these states are bound entangled. In particular, the authors argued that these states are bound entangled, because the states are separable against bipartite symmetric cuts like $12|34\ldots, 14|23\ldots, \ldots$ and therefore no Bell state between any two subsystem can be distilled. This is obviously also the case for the whole class of states under investigation.

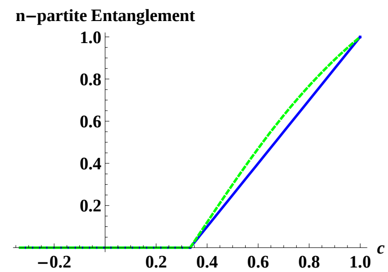

FIG. 2: (Color online) Here the n–partite entanglement of the Werner states $n = 2$ (here the y–axis has to be multiplied by two) or the generalized Smolin states $n > 2$ before and after the application of the introduced protocol (upper dashed green curve) is plotted. Note that the vertex states are mapped onto itself by the given protocol.

As the considered measure of entanglement revealed only n–partite entanglement and e.g. not any m–partite entanglement ($m < n$), it may not seem directly obvious that Bell states (bipartite entanglement) cannot be distilled, because the class of states do not possess any bipartite entanglement. Thus the question could be refined to ask whether n–partite pure entanglement can be distilled.

For the n–partite class of states under investigation we consider a similar distillation protocol as the recurrence protocol by Bennett et al. [22]. For that we generalize it such that each party gets a copy onto which a unitary bilateral XOR operation is performed and afterwards a measurement in say z-direction is performed. Only states are kept where all parties found their copy qubit in say up–direction. This protocol favours as all protocols do one state, in our case for $n = 2$ it is the Φ^+ state and for $n > 2$ its equivalents.

In detail it goes like the following: We consider one state and its copy

$$\rho^{\otimes 2} = \left(\frac{1}{2^n}\left\{\mathbb{1}^{\otimes n} + c_i\, \sigma_i^{\otimes n}\right\}\right)^{\otimes 2} \qquad (23)$$

and all parties get a copy state. Therefore, we reorder the state by a unitary transformation such that the first term and second term in the tensor product belongs to Alice and the third and forth term to Bob and so on:

$$\begin{aligned}\rho^{\otimes 2} \longrightarrow (\frac{1}{2^n})^2 \Big\{ (\mathbb{1} \otimes \mathbb{1})^{\otimes n} + c_i\, (\mathbb{1} \otimes \sigma_i)^{\otimes n} \\ + c_i\, (\sigma_i \otimes \mathbb{1})^{\otimes n} + \; c_i c_j\, (\sigma_i \otimes \sigma_j)^{\otimes n} \Big\} \qquad (24)\end{aligned}$$

Now each party perform on its two subsystems a unitary

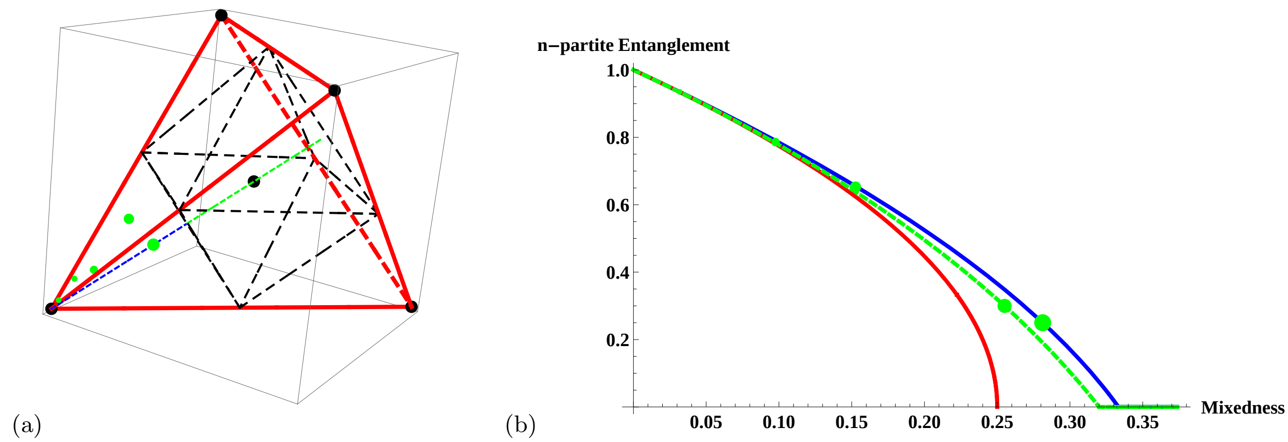

FIG. 3: (Color online) Fig. (a) shows the final states after each step of the introduced protocol of an initial Werner or Smolin state $c = 0.5$, where each (green) point represents the obtained state after one step of the protocol. Fig. (b) shows a mixedness, $\frac{2^n}{2^n-1}(1 - Tr(\rho^2))$, versus n–partite entanglement diagram (for $n = 2$ the y–axis has to be multiplied by 2), where the (blue) curve corresponds to the Werner or Smolin state whereas the (red) curve is the state connecting two vertices. All states of the simplex have their mixedness–entanglement ratio between these two curves. The middle (dashed, green) curve corresponds to the final states of a distilled Werner or Smolin state. And the (green) points represent the final states after each step of an initial Werner or Smolin state $c = 0.5$.

XOR operation

$$U_{XOR} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{25}$$

and then projects on the copy–subsystem with $P = \frac{1}{2}(\mathbb{1} + \sigma_z)$. This gives again a state in the class of states under investigation, i.e. one finds

$$\vec{c} = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} \longrightarrow \vec{c}_{\text{dis}} = \begin{pmatrix} \frac{c_x^2 + c_y^2}{1+c_z^2} \\ \frac{2c_x c_y}{1+c_z^2} \\ \frac{2c_z}{1+c_z^2} \end{pmatrix} . \tag{26}$$

Comparing with the separability condition and with the positivity condition, one verifies that only separable states are mapped into separable states.

Let us consider the Werner states and the generalized Smolin states ($c = c_x = c_y = c_z$), for which we derive that the n–partite entanglement is always increased after the above protocol, see Fig. 2. For $-\frac{1}{\sqrt{3}} \leq c \leq \frac{1}{3}$ the measure before and after the protocol is zero and for $c = 1$ the state is mapped onto itself. For $\frac{1}{3} < c < 1$ the entanglement of the distilled state is increased compared to the input state. In Fig. 3 (a) we give the 3–dimensional picture of how the initial state $c = 0.5$ moves after each step towards the vertex state. Note that the states are no longer in the set of the generalized Smolin sets, another advantage of considered set of states as no random bilateral rotation to regain the rotational symmetry is needed. In Fig. 3 (b) we show the mixedness–entanglement relation of this example. Note that all states of the simplex are within the two curves and the middle curve is the result for the generalized Smolin state after one step of the protocol.

Remark: Not all states of the simplex are mapped into more entangled states by this protocol. For example, the mixture of two vertex states ($\vec{c}^T = (0, 0, c)$ with $c \neq 1$) is left invariant.

In summary, we have found a protocol that increases the amount of entanglement with local operations and classical communication only and the final states are always within the class of states. Only for $n = 2$ the final state is pure and maximal entangled and therefore the above protocol is a distillation protocol, i.e. pure maximally entangled states can be obtained. However, for $n > 2$ the final state is no longer pure, but has the maximal n–partite entanglement of the class of states under investigation.

Thus the next logical step is to search for a distillation protocol which distills the vertex states into pure maximally entangled states, i.e. GHZ–states. However this is not possible for the following reasons: In general, any equally weighted mixture of two maximally entangled states cannot be distilled by mainly two observations. As for all maximally entangled states ρ_i obviously the entanglement can only be reduced by any completely positive map $\Lambda : \rho_i \mapsto \rho_i'$, i.e. $E(\rho_i') \leq E(\rho_i) \; \forall \, \Lambda$. And as the entanglement $E(\rho)$ is convex, i.e. $E(\rho_i') + E(\rho_j') \leq 2E(\rho_i')$, we conclude that at least one ρ_i must be mapped unitary onto itself or another maximally entangled state. Because all maximally entangled states are equivalent by local unitaries, such a map consequently maps also the other maximally entangled state of the mixture into a (different) maximally entangled state. Hence, for no equally mixture of maximally entangled states a maximally entangled state can be distilled. Note that in the

case of bipartite qubits this is trivially true, because any equally mixture of Bell states is separable, however, for multipartite states this is not necessarily the case (e.g. our vertex states).

Thus we find that we can increase the amount of the n–partite entanglement until the vertex state, but not furthermore and therefore all entangled states are bound entangled, i.e. no pure n–partite entanglement can be distilled among any subset of parties using stochastic LOCC. The common definition of distillation is that no pure maximally entanglement among any subset of parties using LOCC can be obtained, see e.g. [23, 24]. A different way to prove that the entangled states are bound is given in Ref. [25], where they show that if no singlets can be distilled also no GHZ—state can be obtained. Therefore for the class of states under investigation we can also not distill any bipartite entanglement.

VI. THE GEOMETRY OF THE STATES VIOLATING THE CHSH—BELL INEQUALITY

Analog to the bipartite qubit state one can derive a CHSH–Bell type inequality for n qubit states [26]. Here $n-1$ parties measure their qubit in direction $\vec{a}$ or $\vec{a}'$ and the nth party in direction $\vec{b}$ or $\vec{b}'$, then one obtains the following Bell inequality

$$Tr(\mathcal{B}_{\text{Bell-CHSH}}\rho) \;\leq\; 2 \tag{27}$$

with

$$\begin{aligned}\mathcal{B}_{\text{Bell-CHSH}} \;=\; & \underbrace{\vec{a}\vec{\sigma}\otimes\vec{a}\vec{\sigma}\otimes\cdots\otimes\vec{a}\vec{\sigma}}_{n-1}\otimes(\vec{b}+\vec{b}')\vec{\sigma} \\ & + \underbrace{\vec{a}'\vec{\sigma}\otimes\vec{a}'\vec{\sigma}\otimes\cdots\otimes\vec{a}'\vec{\sigma}}_{n-1}\otimes(\vec{b}-\vec{b}')\vec{\sigma}\end{aligned} \tag{28}$$

where $\vec{a},\vec{a}',\vec{b},\vec{b'}$ are real unit vectors and the value 2 is the upper bound on any local realistic theory.

It is known that for $n=2$ the maximal violation by quantum mechanics can simply be derived by the state ρ itself [27]. A matrix ρ violates the Bell–CHSH inequality if and only if $\mathcal{M}(\rho)\geq 1$, where $\mathcal{M}(\rho)$ is the sum of the two largest eigenvalues of the Hermitian matrix $C^\dagger C$ with $(C)_{ij} = Tr(\sigma_i\otimes\sigma_j\rho)$. A generalization for n qubits is simple, because the matrix C is diagonal for the states under investigation, thus the same proof works.

In our case $\mathcal{M}(\rho)$ is simply the sum of the two largest squared vector components. In particular, if c_1 and c_2 are greater than c_3 we obtain the following Bell inequality

$$c_1^2 + c_2^2 \leq 1\;. \tag{29}$$

This gives a simple geometric interpretation of all states violating the Bell inequality. All possible saturated Bell inequalities give three different cylinders in the picture representing the state space, see Fig.4. All states outside of these three cylinders violate the Bell inequality.

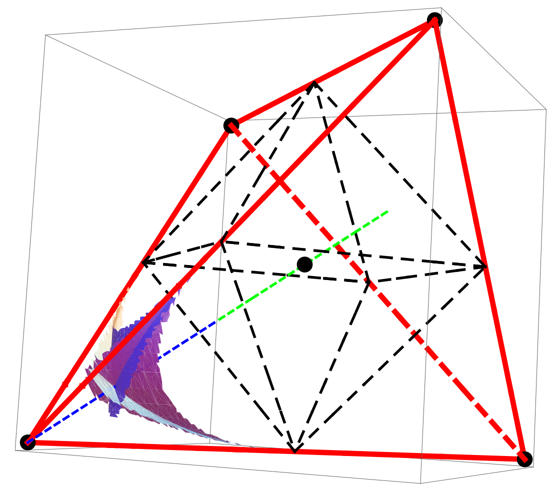

FIG. 4: (Color online) The three cylinders show the saturation of the Bell inequality. All states outside these cylinders violate the Bell inequality. The vertex states violate the Bell inequality maximal, i.e. by $2\sqrt{2}$.

Furthermore, this result shows that an entangled state not violating the Bell inequality (27), can be transformed via the introduced protocol into a state violating the Bell inequality, leading to the conclusion that all entangled states of the picture have nonlocal features. Moreover, in agreement with Ref. [28] the possibility to construct realistic local models or not is no criterion for being bound entangled or not.

Let us also remark that Werner states ($n=2$) violate the Bell inequality for $c > \frac{1}{\sqrt{2}}$ whereas successful teleportation requires only $c > \frac{1}{2}$.

VII. SUMMARY AND DISCUSSION

We generalized the magic simplex for locally maximally mixed bipartite qubit states such that we add even numbers n of qubits and set all partial traces equal to the maximally mixed states, i.e. no local information obtainable by any subset of parties is available. This class of states can be described by three real numbers which enables us to draw a three dimensional picture. Interestingly, we find the same geometry concerning separability, mixedness, kind of entanglement, amount of entanglement and nonlocality for all even numbers of qubits, see also Fig. 1 and Fig. 4.

For $n>2$ the purest states, located in the vertices of the simplex, are not pure except in the case of bipartite qubits ($n=2$). We show how to derive a recently proposed measure for all discrete multipartite systems [1] in this case. For mixed states only bounds exist, however, we show that they are for the class of states optimal by

proving that all states not detected by the measure are separable.

The measure reveals that these states only possess n–partite entanglement and no other kind of entanglement, e.g. bipartite entanglement. The information content of the states can be quantified by the generalized Bohr's complementarity relation for $n > 2$

$$n \;=\; \mathcal{S} + E_n + R \;=\; 1 + R\,, \tag{30}$$

where R lack of classical knowledge and $\mathcal{S} = 0$ the local information obtainable by any party.

Then we investigated the question whether the n–partite entanglement can be distilled. We find a protocol using only local operation and classical communication (LOCC) which increases the n–partite entanglement to the maximal entanglement of the class of states under investigation. These states are the vertex states of the simplex, for $n = 2$ they are the Bell states and for $n > 2$ they are equal mixtures of such GHZ–states which are obtained by applying only two flips, σ_x.

For bipartite qubits $n = 2$ this protocol is a distillation protocol, i.e. pure maximally entangled states are obtained. For $n > 2$ the vertex states are not pure, therefore we search for a distillation protocol that leaves the class of states under investigation to obtain a pure n–partite maximally entangled state, i.e. the GHZ–states. Indeed, we argue that such a protocol cannot be found, more precisely, any equal mixture of GHZ–states cannot be distilled. Thus for the class of states under investigation all entangled states are bound entangled and herewith we found a simplex where all states are either separable or bound entangled.

In detail, we show how an initial state moves after each step of the protocol increasing the entanglement in the simplex, see Fig. 2. Moreover, we find that the states violating the CHSH–Bell like inequality, which was shown to be optimal in this case, have for all even numbers of qubits the same geometry, see Fig. 4. These two results taken together mean that one can enhance the n–partite bound entanglement by only using LOCC until the Bell inequality is violated. Therefore, for all n–partite bound entangled states its (hidden) nonlocality is revealed and in agreement with Ref. [28] a possibility whether a local realistic theory can be constructed is not a criterion for distillability and likewise whether its entanglement can be increased by LOCC is also no criterion.

Our results suggest also that one can distinguish between bound states for which a certain entanglement measure cannot be increased by LOCC (in our case the vertex states) and states for which the entanglement can be increased by LOCC, which may be denoted by "quasi" bound entangled states (all bound entangled states of the class except the vertex states). The introduced (distillation) protocol distills maximally entangled states within the set of states which are, however, not pure, but the purest of the set of states.

Last but not least we want to remark that a subset of the class of states was considered in literature, e.g. [4, 6, 7, 8], the so called Smolin states. For which it was shown that no Bell states may be distilled. The theorem in Ref. [25] states that if and only if bipartite entanglement can be distilled then also GHZ-states —in our terminology n–partite entanglement— can be distilled.

In summary, we have shown in this paper explicitly that the multipartite measure proposed by [1] detects all bound entanglement in the class of states and that the states do not possess bipartite entanglement and how the n–partite entanglement can be increased to a certain value.

These results do not only help to reveal the mysteries of bound entanglement by refining its kind of entanglement, but they may also help to construct quantum communication scenarios where bound entangled states actually help to perform a certain process [29]. This is clearly important, when one has future application in mind, e.g. a multipartite cryptography scenario.

Acknowledgement: Many thanks to B. Baumgartner, R.A. Bertlmann, W. Dür and R. Augusiak for enlightening discussions.

Appendix: Proof that all states represented by the octahedron are separable.

To prove that all states represented by the octahedron are separable, we show that this is the case for the following points in the octahedron

$$\vec{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\;. \tag{31}$$

As any convex combination of separable states have to be also separable, we have finalized the proof. We start with $n = 2$ and show how this construction generalizes for $n = 4, 6, \ldots$.

Suppose Alice prepares her qubits in the following two states:

$$\omega^A_{i,\pm} \;=\; \frac{1}{\sqrt{2}}(\mathbb{1}_2 \pm r^A_i \sigma_i)\,, \tag{32}$$

where r_i is a Bloch vector pointing in i–direction and is given by any number in $[-1, 1]$. Bob does prepares his qubits in the very same way. If Alices chooses the positive i–axis and Bob does the same, if Alice chooses the negative sign, Bob does the same, thus they share the following separable state if the preparation is done randomly with the same probability:

$$\begin{aligned} \rho^{AB}_{i,+} \;&=\; \frac{1}{2}\,\omega^A_{i,+}\otimes\omega^B_{i,+} + \frac{1}{2}\,\omega^A_{i,-}\otimes\omega^B_{i,-} \\ &=\; \frac{1}{4}(\mathbb{1}_4 + r^A_i\cdot r^B_i\;\sigma_i\otimes\sigma_i)\,. \end{aligned} \tag{33}$$

These states represent three vertices of the octahedron, thus the proof is finalized for $n = 2$.

Explicitly, we find that for the generalized Smolin state ($c_1 = c_2 = c_3 = c$), the following state derives

$$\rho_c \;=\; \sum_i \frac{1}{3}\,\rho^{AB}_{i,+} \;=\; \frac{1}{4}(\mathbb{1}_4 + \sum_i \frac{r^A_i\cdot r^B_i}{3}\;\sigma_i\otimes\sigma_i)\,. \tag{34}$$

therefore as $r_i^A \cdot r_i^B \in [-1,1]$ the generalized Smolin state is separable for $p \in [-\frac{1}{3}, \frac{1}{3}]$.

For $n = 4$ we remark that with the combination

$$\begin{aligned} \rho_{i,-}^{AB} &= \frac{1}{2}\omega_{i,+}^A \otimes \omega_{i,-}^B + \frac{1}{2}\omega_{i,-}^A \otimes \omega_{i,+}^B \\ &= \frac{1}{4}(\mathbb{1}_4 - r_i^A \cdot r_i^B \; \sigma_i \otimes \sigma_i) \end{aligned} \tag{35}$$

one obtains the minus sign, and for the very same construction Alice, Bob, Charly and Daisy obtain the following separable states

$$\begin{aligned} \rho_{i,+}^{AB} &= \frac{1}{2}\, \rho_{i,+}^{AB} \otimes \rho_{i,+}^{CD} + \frac{1}{2}\, \rho_{i,-}^{AB} \otimes \rho_{i,-}^{CD} \\ &= \frac{1}{4}(\mathbb{1}_4 + r_i^A \cdot r_i^B \cdot r_i^C \cdot r_i^D \; \sigma_i \otimes \sigma_i \otimes \sigma_i \otimes \sigma_i) \end{aligned} \tag{36}$$

As the combination $+-, -+$ gives again the minus sign this proof generalizes for any even n.

Appendix: All 4–flip operators for $n = 4$: For convenience of the reader we list all 4–flip operators in the Pauli–matrix representation:

$$\begin{aligned} \mathcal{O}^1 &= \frac{1}{4}\{ \;\; \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\ &\quad - \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \\ &\quad - \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x \\ &\quad - \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y \} \\ \mathcal{O}^2 &= \frac{1}{4}\{ \;\; \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\ &\quad - \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \\ &\quad + \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x \\ &\quad + \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y \} \\ \mathcal{O}^3 &= \frac{1}{4}\{ \;\; \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\ &\quad + \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \\ &\quad - \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x \\ &\quad + \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y \} \\ \mathcal{O}^4 &= \frac{1}{4}\{ \;\; \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\ &\quad + \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \\ &\quad + \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x \\ &\quad - \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y \} \end{aligned} \tag{37}$$

$$\begin{aligned} \mathcal{O}^5 &= \frac{1}{4}\{ \;\; \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\ &\quad - \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \\ &\quad - \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\ &\quad - \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \} \\ \mathcal{O}^6 &= \frac{1}{4}\{ \;\; \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\ &\quad - \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \\ &\quad + \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\ &\quad + \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \} \\ \mathcal{O}^7 &= \frac{1}{4}\{ \;\; \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\ &\quad + \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \\ &\quad - \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\ &\quad + \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \} \\ \mathcal{O}^8 &= \frac{1}{4}\{ \;\; \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\ &\quad + \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \\ &\quad + \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\ &\quad - \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \} \end{aligned} \tag{38}$$

$$\begin{aligned} \mathcal{O}^9 &= \frac{1}{4}\{ \;\; \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\ &\quad - \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \\ &\quad - \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\ &\quad - \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x \} \\ \mathcal{O}^{10} &= \frac{1}{4}\{ \;\; \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\ &\quad - \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \\ &\quad + \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\ &\quad + \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x \} \\ \mathcal{O}^{11} &= \frac{1}{4}\{ \;\; \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\ &\quad + \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \\ &\quad - \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\ &\quad + \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x \} \\ \mathcal{O}^{12} &= \frac{1}{4}\{ \;\; \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\ &\quad + \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \\ &\quad + \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\ &\quad - \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x \} \end{aligned} \tag{39}$$

$$\begin{aligned}
\mathcal{O}^{13} = \frac{1}{4}\{ & \quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
& - \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \\
& - \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \\
& - \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y\} \\
\mathcal{O}^{14} = \frac{1}{4}\{ & \quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
& - \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \\
& + \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \\
& + \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y\} \\
\mathcal{O}^{15} = \frac{1}{4}\{ & \quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
& + \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \\
& - \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \\
& + \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y\} \\
\mathcal{O}^{16} = \frac{1}{4}\{ & \quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
& + \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \\
& + \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \\
& - \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y\} \qquad (40)
\end{aligned}$$

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[30] Note that for $n = 4, 8, \ldots$ the mirrored tetrahedron ($\vec{c} \longrightarrow -\vec{c}$) is obtained.